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ABSTRACT

This document is a revision of the 1961 report of the same title, taking into account the many changes which have occurred in school mathematics in the past decade. Four objectives of mathematics training are identified and discussed: (1) understanding of concepts and structure; (2) facility with applications; (3) ability to solve problems; and (4) development of computational skills. Specific course recommendations are made at four levels: (I) elementary school teachers (grades K-6); (II-E) specialist mathematics teachers, coordinators and middle school mathematics teachers (grades 5-8); (II-J) junior high school mathematics teachers (grades 7-9); and (III) high school mathematics teachers (grades 7-12). Appendices give details of the content of the proposed courses, including alternative sequences for Level I. Special features of these courses are an emphasis on mathematics as a unified subject (no separate courses in algebra or geometry are recommended for Level I); a constant reference to the importance of the applications of mathematics; the use of flow charts and computers at various levels; and a second course in geometry for high school teachers, using vectors and transformations. (MM)

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

MATHEMATICAL ASSOCIATION OF AMERICA

RECOMMENDATIONS

ON

COURSE CONTENT

FOR THE

TRAINING OF TEACHERS

OF

MATHEMATICS

AUGUST, 1971

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IN MATHEMATICS

August, 1971

The Committee on the Undergraduate Program in Mathematics is a committee of the Mathematical Association of America charged with making recommendations for the improvement of college and university mathematics curricula at all levels and in all educational areas.

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Additional copies of this report may be obtained without charge from CUPM, Post Office Box 1024, Berkeley, California 94701.

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INTRODUCTION AND HISTORY

CUPM issued its "Recommendations for the Training of Teachers of Mathematics"* in 1961 during the early stages of modern mathematics curriculum reform. In these recommendations the Committee considered five levels of teachers of mathematics. These levels were:

I. Teachers of elementary school mathematics - grades K through 6

II. Teachers of the elements of algebra and geometry

III. Teachers of high school mathematics

IV. Teachers of the elements of calculus, linear algebra, probability, etc.

V. Teachers of college mathematics.

During the years 1961-62 CUPM also published "Course Guides for the Training of Teachers of Elementary School Mathematics"* and "Course Guides for the Training of Teachers of Junior High School and High School Mathematics."*

When it was proposed, the Level I curriculum received widespread attention and approval. It was approved formally by the Mathematical Association of America, and it was endorsed by three conferences held by the National Association of State Directors of Teacher Education and Certification (NASDTEC) and the American Association for the Advancement of Science (AAAS). It formed a part of the "Guidelines for Science and Mathematics in the Preparation Program of Elementary School Teachers," published by NASDTEC-AAAS in 1963.

In the years 1962-1966 CUPM made an intensive effort to explain its proposed Level I program to that part of the educational community especially concerned with the mathematics preparation of elementary teachers. Forty-one conferences were held for this purpose, covering all fifty states. Participants in these conferences represented college mathematics departments and departments of education, state departments of education, and the school systems. At these conferences the details of CUPM proposals were discussed and an effort was made to identify the realistic problems of implementation of the recommendations. A summary of the Level I Conferences is given in [1].**

As a result of these conferences and of other forces for change, there has been a marked increase in the level of mathematics training required for the elementary teacher. In 1966 CUPM repeated

*Now out of print.

**For a bibliography of CUPM publications, see page 22.

a study it had made in 1962 of the graduation requirements in the various colleges having programs for training elementary teachers. A summary of this study is given in [1], but two of its important results are revealed in the following table:

	1962	1966
Per cent of colleges requiring no mathematics of prospective elementary school teachers	22.7	8.1
Per cent of colleges requiring five or more semester hours of mathematics of these students	31.8	51.1

Level II and III Conferences similar to those held for Level I were deemed unnecessary by CUPM in 1962 because the Level II and III guidelines had apparently been accepted by the teaching community through distribution of the recommendations and course guides. One indication of this acceptance has been the publication of numerous textbooks whose prefaces claim adherence to the CUPM guidelines.

Throughout the decade of the 1960's CUPM continued to expend considerable effort on the problems associated with the preparation of teachers. Minor revisions of the original 1961 recommendations were produced in 1966, and the course guides for Level I were similarly revised in 1968. In 1965 CUPM published "A General Curriculum in Mathematics for Colleges" (GCMC)[2] as a model for a mathematics curriculum in a small college. GCMC became a standard reference in other CUPM documents. For instance, in the 1966 revision of the 1961 Teacher Training Recommendations, the original recommendation of a Master's Degree for Level IV preparation was further delineated by the specification that "the program include the equivalent of at least two courses of theoretical analysis in the spirit of the theory of functions of real and complex variables. The courses 11, 12, 13 of the GCMC report are at the proposed level for undergraduate preparation and indicate the sort of material desirable for graduate study."

In 1967 CUPM completed its guidelines for Level V preparation with the report "Qualifications for a College Faculty in Mathematics" [3]. The growing importance of two-year colleges in American higher education led to the publication by CUPM in 1969 of "A Transfer Curriculum in Mathematics for Two Year Colleges" [4], and also in 1969, a companion report entitled "Qualifications for Teaching University Parallel Mathematics Courses in Two Year Colleges" [5].

The publication of these reports completed CUPM's original plan of providing course guides for each of the five teaching levels defined in 1961. By 1967, however, the pressure for further change was beginning to be felt. A minor revision (1968) of the Level I course guides contained the statement, "The five years that have elapsed since the preparation of the Course Guides have seen widespread adop-

tion of the ideas of the new elementary school curricula, not only of the work of such experimental or quasi-experimental groups as the School Mathematics Study Group (SMSG) or the University of Illinois Curriculum Study in Mathematics (UICSM), but also of many new commercial textbook series which incorporate such ideas. In addition, there have been attempts to influence the future direction of elementary school mathematics by such groups as the Cambridge Conference.* In the near future, the Panel believes, it will be necessary to examine our courses to take account of these developments. We hope in the next couple of years to begin the sort of detailed, intellectual study of current trends in the curriculum and of predictions of the future which will be necessary in order to prepare teachers for the school mathematics of the next twenty years."

CUPM's Panel on Teacher Training has, since 1968, continued this promised study. It has sought to understand current trends and future possibilities through a variety of means: in the spring of 1968 it sponsored a conference, New Directions in Mathematics, to obtain the views and advice of a large number of mathematicians and educators; it has constantly followed the deliberations of the CUPM Panel on Computing; it has followed with interest, and has contributed to, continuing discussions on pedagogy, the changing attitudes toward experimentation in mathematics education, and the role of mathematics in society today; and, finally, the Panel has met with representatives of the National Council of Teachers of Mathematics, the American Association for the Advancement of Science, and the National Association of State Directors of Teacher Education and Certification, and has maintained contact with national curriculum planning groups (e.g., SMSG, UICSM, Comprehensive School Mathematics Program, Secondary School Mathematics Curriculum Improvement Study). The Panel concluded from this study that a revision of the CUPM Recommendations and Course Guides for Levels I, II, and III was indeed required. This report is the result of that decision, and it contains, in the appropriate places, the Panel's reasons for the decision.

As indicated by its title, this report, like those that preceded it, deals only with questions of mathematical content, although other aspects of teacher preparation are discussed briefly on page 20. Finally, it was regarded as advisable to publish the new recommendations and course guides for all three levels in a single document.

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*Goals for School Mathematics. The report of the Cambridge Conference on School Mathematics, Houghton Mifflin Company, Boston, Massachusetts, 1963.

THE PURPOSE OF THIS REPORT

This report presents an outline of the Panel's new recommendations for the minimal college preparation for teachers of school mathematics based upon its assessment of those significant changes that have taken place or can be expected to take place in school curricula during the 1970's.

The nature of school mathematics is, of course, far from static, and the forces for change are many. The past twenty-five years have produced a phenomenal increase in the quantity of known mathematics, as well as in the variety and depth of its applications. This growth has been reflected in our total culture, which has become increasingly mathematical, a trend which is certain to continue. It is inevitable and proper that these changes will be reflected in the content of school mathematics, as well as in the way it is taught.

Thus, in the past ten years we have seen a flurry of activity directed toward improving the mathematics curricula in our schools. The pace of change alone demands that those engaged in such activity periodically review their efforts. A decade seems to be an appropriate period for such a review.

It is reasonable to ask what specific changes in mathematics and mathematics education during the past decade impel us to modify our recommendations.

The dependence of western civilization on technology has long been evident, and it has been recognized that mathematics supports the physical and engineering sciences upon which technology thrives. More recently new applications of mathematics to the biological, environmental, and social sciences have developed. Statistics and probability have emerged as important tools in these applications. Indeed, as ordinary citizens we frequently encounter surveys and predictions that make use of probability and statistics, so that an intelligent existence demands our understanding of statistical methods. Consequently, many secondary schools have begun to teach probability and statistics and, as we shall show, there are compelling reasons to teach these subjects in the elementary grades.

But aside from the specific mathematics, e.g., statistics, which is brought to bear upon applications, the applications are interesting in themselves, so that it is pedagogically sound to incorporate them in the mathematics program. Thus, our new recommendations emphasize the applications of mathematics.

Many of these new applications have been aided, perhaps even made possible, by modern computers, and the teaching of computer science and computational mathematics are becoming commonplace in our

colleges.* Computer programming is now a part of many junior and senior high school mathematics programs, and it has been discovered that the notion of a flowchart for describing an algorithmic process is a useful pedagogical device in the teaching of elementary mathematics, as well as an efficient device for prescribing a computer program. Thus, computers are influencing mathematics education at all levels, and we have attempted in preparing this report to assess this influence and to recommend measures for increasing it.

It is a fact that change induces change. For instance, an important aspect of curricular change over the past decade has been emphasis on the understanding of mathematical concepts. As a result, we have learned that abstract concepts can be assimilated at a much earlier age than was previously thought possible. Thus, we are less reluctant today to suggest that elementary notions of probability may be useful in explaining ideas about sets and rational numbers than we were a decade ago to suggest that elementary ideas about sets might be useful in helping children to understand the process of counting. Furthermore, the curricular revisions of the past decade have led to improved training programs which have produced elementary school teachers who are more confident about presenting mathematical topics. It is our purpose in this report to take advantage of today's teachers' new attitudes and skills in order to meet new challenges.

Finally, our recommendations are intended to reflect improved preparation over the past decade of entering college freshmen.

*See Communications of the Association for Computing Machinery, Vol. II, No. 3, March 1968, for "Curriculum 68, Recommendations for Academic Programs in Computer Science," as well as the 1971 CUPM publication, "Recommendations for an Undergraduate Program in Computational Mathematics." [6]

THE OBJECTIVES OF TEACHER TRAINING

The Panel believes that the following objectives of mathematical training are important:

1. Understanding of the concepts, structure, and style of mathematics
2. Facility with its applications
3. Ability to solve mathematical problems
4. Development of computational skills.

These statements deserve amplification.

It is our belief that the disciplined, rational man has the best chance of becoming independent, mature, and creative, and that the development of these qualities is a lifelong process. The intellectual discipline of mathematics contributes in a unique way to this development. We identify two reasons why this is so. First, mathematical concepts are necessarily rooted in man's awareness of the physical world. Understanding mathematics allows him to relate more efficiently to his environment. Second, a person's understanding of a concept depends upon its meaningful relation to and firm grounding in his personal experience, as well as upon his awareness of its role in a system of interrelated ideas. As he learns to relate concepts to one another in an orderly fashion he becomes better organized and he improves his ability to abstract and to generalize, that is, to recognize a concept in a variety of specific examples and to apply this concept in differing contexts. We believe that the study of mathematics can directly benefit this process of personal organization. We therefore regard it as essential that mathematics be taught at all levels in such a way as to emphasize its concepts, structure, and style.

It is possible, of course, to study, to appreciate, and even to practice mathematics by and for itself, but people who can and wish to do this are rare. For most of us an important value of mathematics is its applicability to other scientific disciplines. The recent fruitfulness of mathematics in this regard has already been mentioned, but this is really in the tradition of mathematics, which has repeatedly responded to other disciplines that seek to apply its theories and techniques. It should also be recognized that the sciences in their turn have stimulated the development of new fields of mathematics. Thus, we believe that students of mathematics should acquire an understanding of its wide applicability in various fields, and for this reason, applications should be emphasized in every course.

When we speak of facility with applications we mean the ability to recognize and delineate a mathematical model of a physical, social, biological, or environmental problem. Being able to solve the related mathematical problem is a skill which we also regard as important. Effective mathematics instruction must include develop-

ment of the ability to attack problems by identifying their mathematical setting and then bringing appropriate mathematical knowledge to bear upon their solution.

Finally, computational skill is essential. Without it the student cannot learn to solve mathematical problems, to apply mathematics, or to appreciate even its simplest concepts and structures. Although we normally think of this skill as including speed and accuracy in applying the common algorithms of arithmetic and algebra, we should keep in mind the fact that it also includes the ability to estimate quickly an approximate result of a computation. Each course should not only provide systematic practice in computation but should also inculcate in the student the skill and habit of estimating.

This report recommends courses that we believe prospective teachers should study in order to help them achieve these objectives, both for themselves and for their future students. First we explain briefly why we believe that teachers need much more mathematical education than most of them are now getting, and why their training needs to be of a special kind in certain cases.

As we indicated earlier, there have been recent improvements in the certification requirements for elementary school teachers, but in our opinion they continue to be inadequate or inappropriate in many cases. Some states require a semester or a year of "college mathematics" without indicating what sort of mathematics this should be. These practices appear to be based on the assumption that little or no special training in mathematics is needed to teach in an elementary school. This assumption has always been unrealistic, and in the present context of rapidly changing and expanding curricula it is wholly untenable.

In some elementary schools the rudiments of algebra, informal geometry, probability, and statistics are already being taught in addition to arithmetic. But even if only arithmetic is taught, the teacher needs sound mathematical training because his understanding affects his views and attitudes; and in the classroom, the views and attitudes of the teacher are crucial. An elementary school teacher needs to have a grasp of mathematics that goes well beyond the content and depth of elementary school curricula.

Similarly, a Level II or III teacher's understanding of mathematics must exceed, both in content and depth, the level at which he teaches. Within the next decade it is to be expected that secondary school teachers will be asked to teach material which many of our present teachers have never studied.

We therefore recommend courses for all teachers which will not only ensure that they thoroughly understand the content of the courses they must teach, but will also prepare them to discuss related topics with able and enthusiastic students. These college courses must also

prepare teachers to make intelligent judgments about changes in content, pace, and sequence of mathematics programs for their schools, and to have the flexibility of outlook necessary to adjust to the curriculum changes which will surely take place in the course of their professional careers.

THE NEW RECOMMENDATIONS

These recommendations concern only the preparation of teachers of elementary and secondary school mathematics,* and, whereas in 1961 these teachers were classifed into three groups, we find it convenient to use four classifications:

LEVEL I. <u>Teachers of elementary school mathematics</u> (grades K through 6)

LEVEL II-E. <u>Specialist teachers of elementary school mathematics, coordinators of elementary school mathematics, and teachers of middle school or junior high school mathematics</u> (roughly grades 5 through 8)

LEVEL II-J. <u>Teachers of junior high school mathematics</u> (grades 7 through 9)

LEVEL III. <u>Teachers of high school mathematics</u> (grades 7 through 12).

These classifications are to be taken rather loosely, their interpretation depending upon local conditions of school and curricular organization. It will be noted that the various classifications overlap. This is a deliberate attempt to allow for local variations.

The reader should note that the training for Level I teaching is a separate program, while, except for their Level I content, the curricula for the further levels form a cumulative sequence.

The recommendations of this report are not motivated by a desire to meet the demands of any special program of mathematics education or the goals of any particular planning organization. We consider our recommendations to be appropriate for any teachers of school mathematics, including teachers of low achievers.

LEVEL I RECOMMENDATIONS

We have already stated that the applications of mathematics, the influence of computers, and the changes wrought in the 1960's in the teaching of mathematics prompt us to revise our "Recommendations for the Training of Teachers of Mathematics" at all four levels. At

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*The preparation of teachers of two- and four-year college mathematics is the subject of two other CUPM reports (see [3] and [5]).

Level I in particular we are aware that new teaching strategies designed to facilitate and enrich learning are being adopted or are the subject of experimentation. Such strategies impose on elementary school teachers the necessity of a deeper understanding of the school mathematics curriculum than is required by conventional teaching methods and increase the teacher's need for knowledge of mathematics well beyond the level at which topics are treated in the elementary classroom.

We believe that this deeper understanding can be better achieved if mathematics is taught, and understood, from the earliest stages as a unified subject. The function concept, for instance, should serve as a unifying thread in elementary school mathematics, and elementary intuitive geometry should be taught for its connections with arithmetic as well as for its own sake. The applications of mathematics reflect its unity and offer an opportunity to illustrate its power. For instance, the notion of a finite sample space in probability can be used at a very elementary level to illustrate the idea of a set (of outcomes), and the probability of an event can motivate the need for rational numbers. Simple statistical problems yield practice in computing with both integers and rational numbers as well as in applying probability theory to practical situations. Finally, the use of flowcharts helps to explain the elementary algorithms of arithmetic as well as to prepare the student for later study of computer programming.

Thus, while the development of the number system should remain the core of the elementary school curriculum and of the content of Level I courses, there are other crucial topics which ought to be contained in the Level I sequence. We stress this point by listing these topics in the following recommendations.

Recommendations for Prospective Level I Teachers:

> We propose that the traditional subdivision of courses for prospective elementary school teachers into arithmetic, algebra, and geometry be replaced by an integrated sequence of courses in which the essential interrelations of mathematics, as well as its interactions with other fields, are emphasized. We recommend for all such students a twelve semester-hour sequence that includes development of the following: number systems, algebra, geometry, probability, statistics, functions, mathematical systems, and the role of deductive and inductive reasoning. The recommended sequence is based on at least two years of high school mathematics that includes elementary algebra and geometry.
>
> We further recommend that some teachers in each elementary school have Level II-E preparation. Such teachers will add needed strength to the elementary school's program.

Our suggestion of an integrated course sequence represents a very important change which these recommendations are intended to bring about, but there are certain to be questions on how this can be accomplished. We attempt in the sequel to provide our answer to such questions.

There are many ways in which to organize the appropriate material into integrated course sequences, and we encourage experimentation and diversity. Two possible sequences of four 3-semester hour courses are described in detail in the course guides beginning on page 23. For reference we list their titles here:

Sequence 1	Sequence 2
1. Number and Geometry with Applications I	1. Number Systems and Their Origins
2. Number and Geometry with Applications II	2. Geometry, Measurement, and Probability
3. Mathematical Systems with Applications I	3. Mathematical Systems
4. Mathematical Systems with Applications II	4. Functions

These sequences differ in the ordering of topics and in the degree of integration, yet both conform to our idea of an integrated course sequence. It may be helpful to discuss briefly, without any attempt at being comprehensive or any desire to be prescriptive, the rationale which led us to these integrated sequences.

The dual role of numbers in counting and measuring is systematically exploited in each of the four-course sequences. In one direction we are led to arithmetic, in the other to geometry. The extension of the number system to include negative numbers may then be explained by reference to both the counting and measuring models. The study of rational numbers may likewise be motivated through measurement and counting, and the arithmetic of the rationals finds justification and natural applications in elementary probability theory. Geometrical considerations lead to vector addition on the line and in the plane--and then in space--and the Pythagorean Theorem leads naturally to irrational numbers.

The arithmetic of decimals can be presented as the mathematization of approximation. Also, the algorithms of elementary arithmetic lead naturally to flowcharts and to a study of the role of computers.

Length, area, and volume have computational, foundational, and group-theoretical aspects. Extensions of ideas from two to three dimensions constitute valuable experience in them-

selves, are useful in developing spatial intuition, and also help in understanding the nature of generalization in mathematics.

Graphs of functions of various kinds, theoretical and empirical, may be studied, incorporating intuitive notions of connectedness and smoothness.

The function concept plays an important role in an integrated curriculum. Counting, operations on numbers, measurement, geometric transformations, linear equations, and probability provide many examples of functions, and common characteristics of these examples should be noted. Ideas such as composition of functions and inverse functions may then be introduced and illustrated by algebraic and geometric examples. Of course, detailed formal discussion of the function concept should come only after examples of functions have been mentioned in various contexts in which it is useful to do so.

Similarly, the elementary notions of logic such as logical connectives, negation, and the quantifiers should be treated explicitly only after attention has been called informally to their uses in other mathematical contexts. Indirect proofs and the use of counterexamples arise naturally and may be stressed when the structure of the number systems is examined. However, in the final stages of a prospective elementary teacher's training it is useful to return to logic in a more explicit way for the purpose of summarizing the roles of inductive and deductive reasoning in mathematics and providing examples of deductive systems in geometry, algebra, number theory, or vector spaces.

Finally, the references to algorithms in the foregoing paragraphs emphasize the pervasive role of computing and algorithmic techniques in mathematics and its applications. The use of flowcharts for describing algorithms is becoming commonplace in the elementary school. Moreover, flowcharts are proving to be an important educational tool in teaching elementary and secondary school students to organize their work in problem solving. These ideas should therefore be encountered by a prospective teacher in his mathematics training. We recommend that computing facilities be made available, so that he will also have an opportunity to implement some algorithms and flowcharts on a high-speed computer using some standard computing language.

The course sequence should include many references to applications outside mathematics. This is self-evident for probability theory, but it is important to stress this over the whole spectrum of topics studied. In particular, the function concept itself provides many opportunities to underline the significance of mathematical formulations and methods in our study of the world around us.

At the conclusion of the course sequence the prospective teacher should understand the rational number system and the necessity, if not the method, of enlarging it to the real number system.

He should be familiar with elementary linear geometry in two and three dimensions. In his study of the integers and the rational numbers, he should understand the essential role played by the properties of the addition and multiplication operations and the order relations in justifying and explaining the usual computational algorithms, the factorization theory of whole numbers, and the methods of solution of equations. He should thereby, and through experience with algebraic structures encountered in geometry, acquire an appreciation of the importance of abstraction and generalization in mathematics.

He should know something of the basic concepts and the algebra of probability theory, and he should be able to apply them to simple problems. He should grasp the idea of an algorithmic process and understand a bit about computers and how one programs them. We expect him to appreciate something of the role of mathematics in human thought, in science, and in society. We hope finally that he can learn all this in such a way that he will enjoy mathematics and the teaching of it, and that he will desire to continue to study mathematics.

LEVEL II RECOMMENDATIONS*

In the years since the first set of recommendations was made, dramatic changes have taken place in the mathematics of junior high school. These changes are in depth, as witnessed by greater emphasis on logic and mathematical exposition, and in breadth, as witnessed by the increased amount of geometry and probability. They make it necessary to re-examine the background needed by a teacher at this level. Moreover, the intermediate position of the junior high school requires of teachers at this level an appreciation of the mathematics of the elementary school as well as knowledge of the mathematics of the high school.

Finally, it seems desirable that there be two kinds of teachers in the middle school or the junior high school: those who concentrate on the transition from the elementary school and those who concentrate on the transition to the high school. For this reason we give two sets of recommendations for this level.**

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*"Guidelines for the Preparation of Secondary School Teachers of Mathematics" have also been prepared by the Committee on the Breadth and Depth of the Mathematics Teacher's Preparation in Science and Mathematics of the American Association for the Advancement of Science. For copies write to AAAS, 1515 Massachusetts Avenue, N.W., Washington, D.C. 20005.

**This preparation is called "area of concentration" in some teacher training programs.

LEVEL II-E RECOMMENDATIONS

These recommendations are for students who begin with Level I preparation and pursue further training to qualify them to be either specialist teachers of elementary school mathematics, or coordinators of elementary school mathematics, or teachers of middle school and junior high school mathematics. As stated on page 10, there should be some teachers with Level II-E preparation in each elementary school. The recommended program is:

A. The Level I program. (A student who is already prepared for calculus may omit the course on functions of the second sequence of courses listed on page 11.)

B. An elementary calculus course (e.g., Mathematics 1 as described in [2]). At this level all teachers need an introduction to analysis and an appreciation of the power that calculus provides.

C. Two courses in algebra. The courses in linear and modern algebra are identical to those described under C of the Level III recommendations, pages 17-18.

D. A course in probability and statistics. This course is identical to the first course under D of the Level III recommendations, page 18.

E. Experience with applications of computing. This recommendation is identical to that under F of the Level III recommendations, page 18.

F. One additional elective course. For example, a further course in calculus, geometry, or computing.

LEVEL II-J RECOMMENDATIONS

These recommendations provide a special curriculum for the training of junior high school teachers which is slightly less extensive than that for Level III. The recommended program is:

A. Two courses in elementary calculus (e.g., Mathematics 1 and 2 of [2]). Greater emphasis on calculus is desirable for this level because teachers at the upper level of the junior high school must see where their courses lead.

B. Two courses in algebra. The courses in linear and modern algebra are identical to those described under C of the Level III recommendations, pages 17-18.

C. One course in geometry. Either of the two courses described under E of the Level III recommendations (page 18) will suffice.

D. A course in probability and statistics. This course is identical to the first course described under D of the Level III recommendations, page 18.

E. Experience with applications of computing. This recommendation is identical to that under F of the Level III recommendations, page 18.

F. Review of the content of courses 1 and 2 of Level I (either sequence) through study or audit. There is a problem with the interface between the elementary school and the junior high school. In part this is caused by the fact that, traditionally, junior high school teachers are prepared for secondary school teaching and hence are little aware, at first, of their students' capabilities and preparation. We therefore believe that some sort of orientation to the mathematical content and spirit of the elementary school mathematics program is necessary to equip the Level II-J teacher properly. Two means have been considered to meet this need. One method would be to give an additional course in college to the prospective Level II-J teacher, a course which would be a streamlined version of courses 1 and 2 of the Level I program. On the whole, we prefer this solution although it makes the plan of study rather long. The second method would be to encourage schools to supply the new Level II-J teacher with elementary school texts to read, and to require him to visit classes and to talk with elementary school teachers, especially those in grades four to six. A combination of both of these methods might prove most effective.

G. Two elective courses.

Items A through E supply the bare essentials. Greater breadth and greater depth are both to be desired. In order to give the teacher freedom to pursue his interests, electives are suggested, with further courses in computing, analysis, algebra, and geometry having high priority. Teachers with Level III preparation can meet the requirements listed above by fulfilling the intent of F.

LEVEL III RECOMMENDATIONS

Although the mathematics of the senior high school has not changed as dramatically in the past ten years as has that of the elementary school, yet there are significant directions of change which make new recommendations desirable. These are: (1) a gradual increase in the volume and depth of mathematics taught at the secondary level which brings with it an increased occurrence of calculus

(with the Advanced Placement program), (2) an increasing use of computers in mathematics courses and as an adjunct in other courses, and (3) an increasing realization that applications should play a more significant role.

Our recommendations, while designed primarily to specify minimum requirements for prospective high school teachers, have also been constructed with a view to maintaining, as far as possible, comparability of standards between prospective teachers and prospective entrants to a graduate school with a major in a mathematical science. We want to maintain a freedom of choice for the student to go in either direction. While the program we recommend for prospective teachers will leave the student with a deficiency in analysis and in algebra in order to meet the CUPM recommendations for entry to graduate school (c.f. [7]), the prospective graduate student in mathematics would normally need courses in geometry and in probability and statistics to meet our recommendations for teachers. We regard it as a matter of great importance that a program for teachers should be identical to the one offered to other mathematics majors, except for a few courses peculiarly appropriate to prospective high school teachers.

Before detailing the recommendations, some remarks on the role of applications, the computer, and on the problem of teaching geometry are in order.

Every experienced teacher knows that mathematics must begin at the concrete level before it can proceed to a more theoretical or abstract formulation. It is assumed that topics in the courses under discussion will contain a judicious mixture of motivation, theory, and application. A purely abstract course for teachers would be madness, but a course in calculation with no theory would not be mathematics. In addition to including applications where possible in mathematics courses, there is a need for introducing some specific study of the lore of mathematical model building, in order to provide the framework of ideas within which specific applications can be placed in their proper perspective. The idea of a mathematical model of a "real" situation and the associated techniques and rationale of the model building process have developed as a sort of folk knowledge among mathematicians and users of mathematics, and now an effort is being directed toward making these ideas more explicit and including them in the curriculum. The course Mathematics 10 described in [2] was such an effort, but only now are detailed descriptions of such a course appearing (e.g., [8]). As these efforts begin to affect the high school curriculum, where much of the material belongs, it becomes more urgent that the future high school teacher receive appropriate preparation. Conversely, the preparation of teachers to communicate these ideas will accelerate the improved treatment of applications in high schools.

Computers have already had a phenomenal impact on the high school mathematics curriculum in supplementing, and in part replacing, traditional formal methods by algorithmic methods. As access to

digital computers becomes more common, one can expect both the flavor and content of high school mathematics courses to change dramatically. In schools where such facilities are already available, it has become clear that opportunities for experimentation and creative outlet, using the computer as a laboratory device, are within the reach of many students whose mathematical ability, motivation, or background would preclude any comparable experience in a formal mathematical setting. Moreover, it has been found that certain abstract mathematical ideas are understood and appreciated more completely when experience is first obtained through the use of a computer. Algorithmic and numerical techniques should therefore be given strong consideration in all courses in which they are appropriate; and, wherever computing facilities are available, use of the computer should be a routine part of these courses.*

The nature of high school geometry continues to change. Changes over the past decade have mainly been toward remedying the principal defects in Euclid's Elements that are related to the order, separation, and completeness properties of the line, but more recently there has developed an entirely new approach to geometry that links it strongly to algebra. This approach is now finding its way into the high school geometry course. A teacher should be prepared to teach geometry either in the modern Euclidean spirit or from the new algebraic point of view. Thus we are recommending that he take two geometry courses at the college level.

The minimum preparation of high school teachers of mathematics should include:

A. Three courses in calculus. The courses 1, 2 and 4 of [2] are suitable. This recommendation assumes that the student has the necessary prerequisites. It is also desirable to take advantage of the growing role of computers in introducing mathematical concepts.

B. One course in real analysis. Course 11 of [2] would be satisfactory provided that the instructor is aware of the primary interest of his students in teaching.

C. Two courses in algebra. One of these should treat those topics in linear algebra that are essential for the understanding of geometry and that have become crucial in applications, especially to the social sciences. Course 3 described in [2], with careful attention to examples, would suffice. The second algebra course should be an abstract algebra course approximating course 6 of [2]. Again, opportunities should be found to incorporate geometrical ideas that motivate and illustrate

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*A good source on programming and problem solving at the appropriate level is: Hull, T. E. and Day, D. D. F., Computers and Problem Solving, Addison-Wesley (Canada), Ltd., 1970.

various algebraic structures (e.g., groups of symmetries, groups of transformations, rings of functions).

D. Two courses in probability and statistics. The first of these should begin with intuitive notions of probability and statistics derived from the real world. Mathematical model building and the relationship of mathematics to the real world should be considered. Calculus may be required in the latter part of this first course. The second course will treat those additional and more advanced topics normally included in a statistics sequence. In [9] the Statistics Panel of CUPM has described two courses that are close to what we have in mind. Throughout the two courses, care should be taken to include an analysis of some statistical studies which have appeared on the public scene and should make explicit some of the misinterpretations that are possible. Applications (in particular, applications to decision theory) should be drawn from such fields as medicine, education, business and politics. The range and realism of problems can be enhanced if students are able to use computers.

E. Two courses in geometry. One course emphasizes a traditional approach by concentrating on synthetic methods and a careful study of the foundations of Euclidean geometry with a brief treatment of non-Euclidean geometry. The other course is strongly linked to linear algebra, includes an investigation of the groups of transformations associated with geometry, and is explicitly related to other parts of mathematics. Examples of such courses are offered on pages 47-55.

F. Experience with applications of computing. This should involve learning the use of at least one higher level programming language such as BASIC or FORTRAN. For this purpose we recommend a formal course such as C-1 of [6], but the experience may also be obtained independently or in other courses that make use of computers.

G. One course in applications. This should place heavy emphasis on mathematical models in the physical or social sciences. CUPM's Panel on Applied Mathematics is presently preparing course outlines for several options for GCMC's course 10, e.g., mathematical models in physical sciences, graph theory and combinatorics, and optimization (see [8]). A course based on examples such as those set forth in "Applications of Mathematics for High School Teachers" by H. Pollak and G. Young [to appear] would also be appropriate.

Nine of these twelve items should be included in the undergraduate program of every prospective high school teacher, namely A, B, C, F, and one course each from D and E. The remaining courses may be deferred to his post-baccalaureate study, although consideration should be given to including them among the electives in his undergraduate program. A list of possible elective courses is included at

the end of the following paragraph. Were it not for our view that an undergraduate program should permit maximum flexibility in choosing a career and as much latitude as possible for every student to express his own interests in acquiring the proper breadth in his area of concentration, we would have specified that all of the twelve items be included in the undergraduate program of a prospective teacher. Fortunately, it is now becoming commonplace for Level III teachers to continue their mathematical education at the graduate level. Indeed, this is mandatory in many instances through permanent certification requirements or through the salary schedules and policies of individual school systems.

In structuring his undergraduate mathematics program, a student will naturally choose his electives after reflection upon his career goals. If, for example, a prospective high school teacher wishes to pursue graduate study in mathematics, he will necessarily choose additional courses in algebra and analysis beyond those which we have mentioned in our recommendations for Level III teachers. We include below a partial list of electives which would suitably extend our recommendations for the training of high school mathematics teachers.

Real Variables (GCMC 11, 12)
Complex Variables (GCMC 13)
Numerical Analysis (GCMC 8)
Abstract Algebra*
Geometry and Topology
Number Theory
Foundations of Mathematics
Logic and Linguistics

*Here we have in mind courses in Introductory Modern Algebra and in Linear Algebra similar to those which will be described in the commentary on [2]. See the reference to [2].

OTHER ASPECTS OF TEACHER TRAINING

In accordance with our charge, we have made recommendations only on the content of the teacher training curriculum. We are, however, well aware that there are other crucial aspects of a teacher's overall preparation. In discussing these matters we hasten to observe that, just as we do not believe in any sharp distinction between teacher-trainees and other students in respect to the content of their mathematics courses, so we insist that these other aspects are relevant to all mathematics instruction. We believe only that they deserve more emphasis for teacher-trainees than for other mathematics majors.

Communication is of the essence in mathematics, and prospective teachers must pay special attention to all of the ways in which mathematics is most effectively communicated. They should be led to regard mathematics as a creative activity--something which one does rather than merely something which one learns. The active participation of the student in the process of discovering and communicating mathematical ideas is crucial for his real understanding. Courses should be taught in ways that foster active student involvement in the development and presentation of mathematical ideas.

Development of skills in writing and reading and speaking and listening should be an explicit part of teacher training at every stage, and not only in mathematics courses. These, like any other skills, can be developed only through constant and active involvement of the student in practices which exercise these skills. Thus, his regular courses, reading courses, clubs, or seminars should stress opportunities for two-way communication of mathematical ideas.

It is also important for teachers to continue to study and to do mathematics throughout their professional lives. This is closely related to, but goes beyond, the processes of communication mentioned above, for a willingness to grow reflects an enthusiasm that often transcends other skills in communicating mathematics.

Other aspects of communication which are not dealt with in this report are those relating to behavioral objectives and to special teaching methods and aids. While the Panel agrees that these are very important matters, it feels that they demand a much more complex effort and a totally different expertise, and might properly be the subject for another study. An excellent volume, which explores "the educational and psychological problems in the selection, organization and presentation of mathematics materials at all levels from the kindergarten through the high school," is the Sixty-ninth Yearbook of the National Society for the Study of Education, entitled "Mathematics Education."

The relationship of mathematics to other studies is another important matter not touched upon in this report except insofar as we have recommended the study of applications of mathematics. Indeed, we believe that every mathematics teacher should develop skill in other subjects which make use of mathematics.

Finally, we share a widespread concern for the special education of the culturally disadvantaged child and of the child whose achievements, for whatever reason, are below accepted standards. Such children require specially trained teachers. We do not know what form this training should take, but we feel that this is a proper concern of CUPM for the future.

A BIBLIOGRAPHY OF CUPM PUBLICATIONS

1. Forty-one Conferences on the Training of Teachers of Elementary School Mathematics, Report No. 15, June, 1966.

2. A General Curriculum in Mathematics for Colleges, 1965.

   Courses 1, 2, 3, 4, and 6 of this document are currently being revised; a commentary on these courses will be published in 1972. Although the references herein are to the commentary, the reader should refer to the original document until the revision is available.

3. Qualifications for a College Faculty in Mathematics, 1967.

4. A Transfer Curriculum in Mathematics for Two Year Colleges, 1969.

5. Qualifications for Teaching University Parallel Mathematics Courses in Two Year Colleges, 1969

6. Recommendations for an Undergraduate Program in Computational Mathematics, 1971.

7. Preparation for Graduate Study in Mathematics, 1965.

8. Applied Mathematics in the Undergraduate Curriculum, to be published.

9. Preparation for Graduate Work in Statistics, 1971.

The above materials are available free of charge from CUPM, P.O. Box 1024, Berkeley, California 94701.

APPENDIX I. COURSE GUIDES FOR LEVEL I

INTRODUCTION

Two sequences, each consisting of four courses, are outlined here in detail. One reason for presenting two different sequences is to illustrate our earlier claim that there are various ways of organizing the material. We have no desire to be prescriptive or definitive with respect either to course content or to the ordering of material. The outlines are to be construed as models only of the content and depth of coverage that we believe will be possible in the best circumstances. We do believe that the material presented approximates that which a really first-rate teacher of elementary school mathematics should know.

We regard either sequence as a way of achieving an integrated curriculum. In the first sequence the courses do not emphasize any particular single area of the traditional curriculum. Thus, arithmetic and geometry are both developed throughout the entire sequence. Each figures prominently in all four courses. In the second sequence, on the other hand, the emphasis is on number systems in the first course, geometry in the second, mathematical systems and induction in the third, and functions in the fourth. In both sequences each course contains topics from most of the areas identified as essential in the recommendation on page 10. By the end of the second course in either sequence the student will have met most of the topics that we consider essential for the elementary teacher, although not at the depth or in the detail preferred. Indeed, throughout both sequences the reader must be careful to interpret the statements of topics to be covered as referring to a treatment appropriate to the level of the student, and not a definitive treatment such as would be accorded to such a topic if encountered at a higher level. Typical places where there is danger of misinterpretation are Section 5 of Course 2 in the first sequence (Operational Systems and Algebraic Structures) and Section 1 of Course 2 in the second sequence (Intuitive Non-metric Geometry).

The unification of the four courses of each sequence requires the use of a common language for the expression of mathematical ideas. For instance, the concepts of set, function, and operation are introduced early and used throughout. Logical terms are introduced and used where appropriate.

Each section of a course guide has a suggested time allocation stated as a percentage of the course. These time allocations indicate first the balance of the sections within the course, second the depth and detail of treatment of the topics listed under each section heading. Thus they should enable the reader to judge the level of treatment and avoid the danger, already referred to, of giving a more comprehensive (or, perhaps, more superficial) treatment than intended.

It must be kept in mind that a prospective teacher is profoundly influenced by what he observes and experiences as a student. Later his own methods and philosophy of teaching will reflect that experience. Hence, it is of paramount importance that these courses be conducted in a manner which encourages active participation in mathematical discovery. Frequent and substantial assignments which expose and drive home the attendant manipulative and computational skills should also be the rule.

SEQUENCE 1

Sequence 1 consists of the four courses

1. Number and Geometry with Applications I
2. Number and Geometry with Applications II
3. Mathematical Systems with Applications I
4. Mathematical Systems with Applications II.

Course 1 begins with some intuitive geometry so that the concept of a number line is available immediately. Then arithmetic and geometry are developed throughout the entire sequence; the interaction of these two areas of mathematics enriches both subjects. Nevertheless, there are occasions when each area is developed within its own context. In particular, the algebra of the rational numbers is applied extensively to the theory of probability and statistics without reference to the geometric aspects of the number line.

A feature of this sequence is the adoption, to a limited extent, of the spiral approach. Thus, certain notions, such as extensions of the number system and the group of rigid motions, reappear several times, each time at a higher level of sophistication and with enhanced mathematical knowledge at the disposal of the student. An important practical advantage of this approach is that the student who takes only two or three courses of the sequence will have met most of the important mathematical concepts. In such cases, however, the depth of understanding is less than desired.

Some simple logic is introduced where appropriate to enhance the student's understanding. At the end of the sequence the student should understand the nature of mathematical reasoning and proof.

Again, we stress the importance of interpreting the Course Guides in the spirit of the comments made in the Introduction, pages 23-24.

Course 1: Number and Geometry with Applications I

1. Elementary Ideas of Space, Measurement, and the Number Line (20%)
2. The Rational Number System and Subsystems (60%)
3. Probability, Statistics, and Other Applications (20%)

Course 1 is concerned with the study of the rational number system. It commences with the most intuitive geometrical notions, from which attention is focused on the number line and its role as a representation of the set of whole numbers. This approach enables the arithmetical and geometrical aspects of elementary mathematics to be developed from an integrated standpoint emphasizing their complementarity. It is a particular feature of the number line that negative integers are thereby immediately suggested and readily studied; this ensures that the development of the number system follows a path which is mathematically natural. The twin approach also enriches the scope for interpretation of the operations of arithmetic. At this stage these operations and their properties are motivated through physical models.

Applications of the arithmetic of the non-negative rationals to the most intuitive ideas of probability and statistics are given. Further applications of the arithmetic should be a feature of the course.

The language should be informal, but it should be such that a transition to precise mathematical language can be naturally effected. In particular, the student should be prepared for the function concept through the use of appropriate language. Those students who have not already met the notions of sets and functions may require more explicit introductions to these concepts.

1. Elementary Ideas of Space, Measurement, and the Number Line (20%)

Intuitive development of geometric figures in the plane and space, first as idealizations of familiar objects and then as sets of points; an intuitive development of incidence relations and some simple consequences; congruence developed by use of slides and flips of models of figures leading to turns as another means of preserving congruence and with attention also to parallelism, perpendicularity, and symmetry; consideration of measurement of segments with various units and the beginning notion of approximation; informal introduction of the number line.

2. The Rational Number System and Subsystems (60%)

Introduction of the set $W = \{0, 1, 2, 3, \ldots\}$* of whole numbers, from sets of objects; addition in W from disjoint union, multi-

*A glossary of symbols is included on page 46.

plication in W from Cartesian products (treated informally); counting as the link between sets and numbers; place value systems and decimal numeration of whole numbers (reinforced by examples with non-decimal bases); properties of operations in W from observed properties of operations on sets, including order properties; algorithms for computation in W (include use of flowcharts); simple closed and open mathematical sentences, including inequalities.

Coordinatization of the half-line with W; addition in W as a vector sum, using slides of the number line; subtraction in W as a slide to the left to introduce the set Z of integers; properties of addition and order properties in Z; mathematical sentences in Z. (Multiplication by negatives is delayed until the set of rationals is developed.)

Factorization in W, prime factorization, unique prime factorization, some simple divisibility criteria, the Euclidean algorithm; multiplication and division of integers by whole numbers introduced through experiences with the number line; greatest common divisor and least common multiple.

Introduction of the set Q of rationals through division by n, $n \in W$ and $n \neq 0$, as shown on the number line; change in scale of the number line and its use in measurement; equivalence classes of symbols for rationals; coordinatization of the line with Q; introduction to the question of completeness.

Addition in Q suggested by slides of the number line; multiplication of a rational by a whole number suggested by slides of the number line; multiplication by a positive rational suggested by stretching and shrinking; multiplication by a negative integer suggested by multiplication by a whole number followed by a flip; multiplication in Q; algorithms for computation in Q^+, properties of addition and multiplication in Q, and order properties in Q.

Decimal numeration of Q; percentages; integer exponents; scientific notation; orders of magnitude; algorithms and flowcharts for computation in Q; mathematical sentences.

3. Probability, Statistics, and Other Applications (20%)

Examples of statistical experiments in finite event spaces and their outcome sets, leading to counting procedures for determining the number of outcomes of various kinds of compound events (use tree diagrams); sampling problems with and without replacements, leading to combinatorial devices for counting samples; relative frequencies of events in an experiment and stability of relative frequencies; assignment of probabilities to singleton events and to disjoint unions and intersections of events through addition and multiplication in Q^+.

Other applications, e.g., measurement, constant rate, profit and loss, expectation and risk, percentages, estimation, significant figures, and approximation.

Course 2: Number and Geometry with Applications II

1. Functions (5%)
2. The Rational Number System and Subsystems (20%)
3. Geometry (35%)
4. Real Numbers and Geometry (10%)
5. Operational Systems and Algebraic Structures (10%)
6. Probability, Statistics, and Other Applications (20%)

Course 2 is designed to give a genuine mathematical treatment of ideas introduced and studied at a more intuitive level in Course 1. The language of functions is established; this enables the solution of linear equations over Q and Z to be investigated systematically. Then the interrelationship between geometry and algebra again becomes evident in the study of symmetries, rigid motions, and sets with operations. At the same time, questions connected with measurement are studied, thus ensuring that the material can be usefully applied; explicit reference to problems of approximate calculation involving large amounts of data can lead to consideration of computer programs.

The Pythagorean relation prepares the way for the introduction of irrational numbers and a preliminary discussion of real numbers. The ideas here are difficult, and no attempt should be made to give complete proofs; nevertheless, the topic should be explored extensively.

Algebraic structures are defined, but studied only in familiar examples (including modular arithmetic). Further study of probability and statistics is included, beginning with a study of permutations and combinations which employs the function concept and presents systematic counting procedures.

1. Functions (5%)

The function concept (motivated by examples from Course 1), one-one and onto properties of functions; relations (motivated by examples from Course 1) with emphasis on equivalence and order relations. Binary operations as functions on Cartesian products of form X x X. Power sets; unions, intersections and complements as operations; the functions $2^X \to 2^Y$ and $2^Y \to 2^X$ induced by a function $X \to Y$.

2. The Rational Number System and Subsystems (20%)

Review of properties of Q with some arithmetical proofs; properties of Z; Z as an ordered integral domain; realization that

Z is not closed under division, with the closure of Z leading to Q.

Coordinatization of the line with Z and then with Q; coordinatization of the plane with Z^2 and then Q^2; and coordinatization of space with Z^3 and Q^3.

3. <u>Geometry</u> (35%)

Review of geometric figures as idealizations of familiar objects and as sets of points in space; review of rigid motions and symmetry; review of congruence, parallelism, and perpendicularity. Groups of symmetries of an equilateral triangle and a square.

Rigid motions as functions that preserve lengths of segments; classification of rigid motions; composition and inverses; intuitive understanding of group properties of the group of rigid motions.

Review of measurement of segments with various units; principles of measurement; principles of measurement applied to length, area, angle measurement, volume; approximation; formulas for measurement related to rectangles, triangles, right prisms, pyramids. Approximate calculation. Pythagorean relation through the formula for the area of a rectangle.

4. <u>Real Numbers and Geometry</u> (10%)

Adequacy of Q for physical measurement; inadequacy of Q for representing lengths of segments demonstrated by construction of segments with irrational measures; distance between points in Q^2 (use of Pythagorean relation). Introduction of real numbers in terms of nested intervals and non-terminating decimals; location of irrational points on the number line; infinite decimals regarded as sequences of approximating rationals; informal definition of addition and multiplication in R by means of approximating terminating decimals; distance in R^2; use of unique factorization to prove the irrationality of $\sqrt{2}$, $\sqrt{3}$, ...

Coordinatization of the line, plane, and space with the real numbers; distance in R^2. Solution of linear equations and inequalities and graphs of solution sets in R^2; solution of linear equations in R^3 as intersections of planes.

Intuitive treatment of the perimeter and area of a circle; π as an irrational number.

5. <u>Operational Systems and Algebraic Structures</u> (10%)

Review of properties of the operations of addition and multiplication and the order relation in the number systems W, Z, and Q;

definitions of group, ring, integral domain, and field, with examples drawn from subsets of Q, groups of rigid motions, groups of symmetries of a figure, power sets.

Arithmetic mod m as an operational system; contrasted with W, Z, and Q (closed under additive inverses, closed under multiplicative inverses when m is prime, divisors of zero when m is not prime, absence of compatible order relation); applications to the arithmetic of W (e.g., Fermat's theorem, Wilson's theorem, casting out 9's).

6. Probability, Statistics, and Other Applications (20%)

Permutations and combinations; randomness of a sample and tests of randomness; examples of applications of random sampling (using random number tables) to estimation of populations, quality control, etc.

Measures of central tendency in lists of data and possible measures of spread of data.

Random walks and their applications; assigning probabilities to compound events with applications; conditional probability.

Other applications, e.g., area, volume, weight, density; constructing an angle whose measure is $\frac{p}{q}$ times the measure of a given angle.

Course 3: Mathematical Systems with Applications I

1. The Rational Number System (15%)
2. The Real Number System (5%)
3. Geometry (35%)
4. Functions (15%)
5. Mathematical Language and Strategy (15%)
6. Probability, Statistics, and Other Applications (15%)

In Course 3 the process of extending the number system from the whole numbers to the rationals, which has been explained and motivated in previous courses from both geometrical and arithmetical considerations, is carried out as a piece of formal algebra. Polynomials are also studied. Then algebra and geometry each provide examples of small deductive systems to illustrate the nature and power of the axiomatic method. The Pythagorean relation is available in this course, so that Euclidean geometry may be carried out in the coordinate plane. Vector notation and methods are studied, and there is a discussion of the generalization of coordinate geometry to three dimensions.

Algebraic concepts are also exemplified by the study of the group of Euclidean motions and certain subgroups, thereby enriching the notion of subgroup with concrete examples. The trigonometric, exponential, and logarithmic functions are defined and studied in their own right and in view of their applications. There are some explicit discussions in this course on mathematical methods, covering such topics as proof, conjecture, counterexample, and algorithms, together with a review of appropriate logical language. Probability theory is itself developed from an axiomatic standpoint, experience of its practical nature having been gained in previous courses.

1. <u>The Rational Number System</u> (15%)

Formal construction of Z from W and of Q from Z.

Polynomials as functions and forms; Q[x] and Z[x] as rings; the degree of a polynomial; substitution; quadratic equations.

Description of proof by induction in W with examples; application to number-theoretic properties of Z, e.g., divisibility properties, Fundamental Theorem of Arithmetic; Euclidean algorithm; similar applications to Q[x]; remainder theorem.

2. <u>The Real Number System</u> (5%)

Review coordinatization of the line with R; approximation of reals by rationals; addition and multiplication in R through rational approximation; the field of real numbers.

3. <u>Geometry</u> (35%)

Review of coordinate geometry of the real plane; distance; graphs of linear equations in two variables.

Plane vectors from translations; vector addition as the composition of translations; multiplication by scalars; vector equations of a line (with resulting parametric and general form of the equations); conditions for parallelism and perpendicularity in coordinate representation; appropriate generalization of these ideas to three dimensions.

Examples of groups and subgroups drawn from geometry, e.g., the group of similarities with the subgroup of rigid motions, the group of rigid motions with the subgroup of rotations about a point, the group of rotations with the subgroup of cyclic permutations of a regular polygon.

Similarities and the representation of similarities as composites of magnifications and rigid motions; similar figures. Constructing the points which separate a segment into n congruent parts.

Equations of circles and the beginning notions of trigonometric functions.

Small deductive system in plane geometry, e.g., incidence properties from postulates of incidence or some constructions from postulates of congruent triangles, or some angle measure properties from postulates of incidence, parallelism, and segment and angle measures.

4. Functions (15%)

Review of real-valued functions and their graphs; inverses of invertible functions; graph of an invertible function and its inverse.

Beginning notions of exponential functions and some of their properties; applications of these ideas, e.g., growth and decay; logarithmic functions; application of logarithmic functions to approximate calculation and construction of a slide rule.

Definitions and graphs of the trigonometric functions with emphasis on periodicity.

5. Mathematical Language and Strategy (15%)

Review of the language of connectives, the common tautologies, and the relation of some of these notions to set union, intersection, complementation, and inclusion.

Universal and existential quantifiers; denial of a mathematical statement, counterexamples.

Examples from previous sections of direct and indirect proof and of proof by induction; selected new topics to illustrate proof by induction (e.g., binomial theorem for positive integer exponents; number of zeros of a polynomial; sums of finite series); explicit contrast with inductive inference, role of hypothesis, conjecture.

Examples of algorithms and flowcharts.

6. Probability, Statistics, and Other Applications (15%)

Postulates for a discrete probability function and some consequences proved for probabilities of compound events; random walks and their applications.

Computation of measures of central tendency and variance; simple intuitive notions of statistical inference, tests of significance, frequency distributions, passage from discrete to continuous variables, normal distribution; application of statistical inference to real life situations, e.g., opinion polls, actuarial tables, health hazards.

Course 4: Mathematical Systems with Applications II

1. Geometry (30%)
2. The Real and Complex Number Systems (10%)
3. Operational Systems and Algebraic Structures (40%)
4. Probability, Statistics, and Other Applications (20%)

Course 4 consists of a systematic study of pre-calculus mathematics. Linear algebra in R^2 and R^3, as vector spaces and as inner product spaces, leads on the one hand to matrix algebra and on the other to the standard trigonometric identities. The algebraic method in geometry is contrasted with the synthetic method. The real numbers R are presented as the completion of the rationals, and the extension of R to the field C of complex numbers is motivated and described.

Abstract algebra occurs in the course--a beginning study is made of abstract group theory--but emphasis is on familiar examples of the various algebraic systems, for example, the integral domain of polynomials over Z, Q, Z_p (p prime). The key notion of homomorphism of algebraic structures is introduced; among the examples treated are the logarithmic and exponential functions which are seen to be mutually inverse isomorphisms.

Frequency distributions form the main topic of the probability and statistics component; although the course remains essentially concerned with discrete probability spaces, the normal distribution is mentioned here. Applications of the preceding theory are made to problems of approximation and error.

1. Geometry (30%)

Coordinatization of space with R^3, distance in space, first degree linear equations in three variables; vectors in space, vector addition, scalar multiples of vectors in R^2 and R^3, description of the vector spaces R^2 and R^3; norms of vectors, inner product, definition of the inner product (or Euclidean) space R^3, relation of cosine to the inner product; definition of the vector product in R^3, triple scalar product and volume; the idea of closeness in R^3, with some of the simpler topological properties of this metric space.

Definition of linear transformations of R^2 and R^3; orthogonal transformations; matrix representation of linear transformations of R^2 and conditions for orthogonality; matrix multiplication suggested by composition of linear transformations; representations of rotations in the plane by orthogonal matrices, leading to the standard trigonometric identities.

Invertible linear transformations of the plane with coordinate representations; rigid motions, magnifications, and other subgroups of the group of invertible linear transformations; representation of similarities in R^2 by matrices.

Analysis of the roles of synthetic and analytic methods in geometry, e.g., properties of circles, coincidence properties of triangles.

2. The Real and Complex Number Systems (10%)

Algebraic extensions of Q; algebraic and order properties of R; discussion of the completeness of R.

Extension of the real number system to the field C of complex numbers; failure of order relations in C; graphical representation of C in R^2.

3. Operational Systems and Algebraic Structures (40%)

System of polynomial forms over Q, its integral domain properties, factorization; Euclidean algorithm and appropriate flow diagram; factor theorem; elementary theory of polynomial equations; comparison with theory for polynomials over Z_p (p prime), Z.

Exponents, extension of exponential functions over Q to functions over R, with graphs; rational functions over Q, over Z_p (p prime); Newton's method of approximating zeros of polynomials (no differential calculus) and appropriate flow diagram.

Subgroups; Lagrange's theorem; applications to elementary number theory (Fermat's theorem and Euler's theorem); commutative groups, quotient groups of commutative groups; application to Z_n.

Homomorphisms of algebraic structures with many examples; identification of those which are one-one, onto; definition of isomorphism as invertible homomorphism; one-one and onto homomorphisms are isomorphisms; isomorphic systems, e.g., Z_4 and rotational symmetries of a square, the positive reals under multiplication and the real numbers under addition.

4. Probability, Statistics, and Other Applications (20%)

Review of sample spaces, probability functions, random walks; discrete binomial distributions; statistical inference and tests of significance; other frequency distributions with applications, e.g., rectangular, Poisson; normal distribution (treated descriptively).

Application of the number system Q to problems in scaling, ratio, proportion, variation; approximation, errors in approximation, errors in sums, errors in products; Bayesian inference.

SEQUENCE 2

Sequence 2 consists of the four courses

1. Number Systems and Their Origins
2. Geometry, Measurement, and Probability
3. Mathematical Systems
4. Functions.

Each course contains topics from most of the areas identified in the general description of the Level I Recommendations (algebra, the function concept, geometry, mathematical systems, number systems, probability, deductive and inductive reasoning). While interrelationships among these topics are explored in the spirit of an integrated curriculum, each of the four courses, nevertheless, has a special emphasis or focus. The emphasis in the first course is on number systems, the second on geometry, the third on mathematical systems, and the fourth on functions.

Although the special character of each course can be suggested in a few words, it is a mistake to assume that any course is narrowly defined by its title. In fact, by the end of the second course the student will have met the full breadth of topics considered essential for the elementary teacher. He will not, however, have reached the depth of understanding desired.

Again, we stress the importance of interpreting the Course Guides in the spirit of the comments made in the Introduction, pages 23-24.

Course 1: Number Systems and Their Origins

1. Sets and Functions (15%)
2. Whole Numbers (45%)
3. First Look at Positive Rational Numbers (10%)
4. First Look at Integers (5%)
5. The Systems of Integers and Rationals (25%)

Course 1 features integration of arithmetic and algebra with supplementary assistance from geometry. The number line is thought of as a convenient device for representing numbers, order, and operations. Rational numbers are introduced in the context of comparing discrete rather than continuous sets (though brief reference is also made to the rational line). Algebraic similarities and differences between the number systems are emphasized. In particular, the systems of integers and rationals are studied in parallel. Algorithms, flowcharts, and manipulative rules for the various number systems are not only justified by referring to physical or schematic models but also are seen as consequences of the algebraic structural properties of the number systems. The whole number system receives much atten-

tion, as its algebraic properties (and its algorithms) recur in only slightly altered form in the systems of integers, rationals, and reals.

1. Sets and Functions (15%)

Review, at an intuitive level, of the basic concepts associated with sets and functions in order to establish the language and notation that will be used throughout the course. (For most students this will be a review of things they have seen repeatedly since junior high school.) Set concepts covered are: membership, inclusion, and equality for sets; various ways of describing sets (rosters, set-builder notation, Venn diagrams); special subsets that often lead to misunderstanding (empty set, singletons, the universal set); common operations on sets (intersection, union, complementation, Cartesian product); illustration of the above concepts in various ways from real objects and from geometry.

The connection between set operations and logical connectives; for example, "or," "and," and "not" are related to union and intersection and complement, while the inclusion relation "$A \subset B$" is related to the implication "$x \in A \Longrightarrow x \in B$." (In this first introduction of logic, the treatment should be very brief and informal, but the language is necessary for subsequent use.)

The major function concepts to be covered include an intuitive rule-of-assignment definition; various ways of specifying this rule (arrow diagram, table, graph, set of ordered pairs, formula); notions of domain and range; input-machine-output analogy; one-one and onto properties; one-one correspondences between finite sets and between infinite sets; brief look at composition and inverses with a view toward later ties to rational arithmetic. (Real and geometric examples should be used.)

2. Whole Numbers (45%)

Whole numbers are motivated by a desire to specify the "size" of finite sets, numerals and numeration systems by the inadequacy of verbal "symbols" for numbers; the Hindu-Arabic numeration system contrasted with historical and modern artificial numeration systems in order to emphasize the roles of base and place value; order among whole numbers related to the process of counting and to the existence of one-one or onto functions between finite sets; order in W represented schematically by the position of points on a whole number line. In the construction of this "line" the concept of congruent point pairs arises naturally.

The operations of addition and multiplication related, with counting as the link, to the set operations of union and Cartesian product (e.g., the use of multiplication in determining the area of a rectangle); multiplication also related to repeated addition and to

determining the number of outcomes in a multi-stage experiment (the product formulas for C_r^n and P_r^n might be illustrated); addition and multiplication represented schematically in the usual vector fashion (slides and stretches) on the number line.

The algorithms of whole number arithmetic justified initially (as in the elementary classroom) by reference to manipulating and grouping finite sets. (A flow diagram for division via repeated subtraction can be given.) The whole numbers with their operations and order now viewed as a mathematical system, the algebraic properties of which are motivated by reference to finite sets and set operations; the algorithms of whole number arithmetic re-examined from an internal point of view and justified on the basis of notational conventions and fundamental algebraic structural principles. The importance of estimating products and quotients should be emphasized as the algorithms are studied.

3. <u>First Look at Positive Rational Numbers</u> (10%)

Fractions motivated by a desire to compare two finite sets. If a probabilistic flavor is desired, compare a set of favorable outcomes with a set of possible outcomes; if a more conventional approach is desired, "ratio" situations can be used. With fractions viewed as operators, addition continues to correspond, in a sense, to disjoint union, while multiplication corresponds to composition; fractions represented schematically as points or vectors on a number line, and the operations viewed vectorially; rules for manipulating fractions motivated initially from physical or schematic representations. Rational numbers appear as abstractions of equivalence classes of fractions, and some algebraic properties of the system of rational numbers can be motivated by physical examples; the algebraic structure of Q^+ is not explored in detail. (The embedding of W in Q^+ considered briefly with a light touch.) A careful structural investigation deferred until the full system Q appears.

4. <u>First Look at Integers</u> (5%)

Integers suggested by some real situation, e.g., profit-loss, up-down, etc.; addition corresponds to an operation in the given situation; multiplication by a positive integer considered as repeated addition. Again the number line is used as a schematic representation; the various uses of the symbol " - " clarified; some algebraic properties identified and motivated; the embedding of the whole numbers in Z introduced, but only lightly.

5. <u>The Systems of Integers and Rationals</u> (20%)

Following a brief review of the concepts of open sentence, variable, replacement set, truth set, equation, and solution set (or

perhaps only the last two) a systematic, parallel exposition of the algebraic structures of Q and Z in terms of solutions to equations is possible. (-a represents the unique solution to $a + x = 0$, $1/a$ represents the unique solution to $ax = 1$ ($a \neq 0$), $b - a = b + (-a)$, $b/a = b \times 1/a$); all the familiar rules for manipulating minus signs and fractions follow. The two important unique representations of rationals--as fractions in lowest terms and as (all but one type of) repeating decimals--can be illustrated and computational rules for (finite) decimals justified; several non-repeating infinite decimals described. The work on fractions in lowest terms will involve a certain amount of number theory, which should be done on an ad hoc basis. Review of the concepts of divisibility and prime; the Fundamental Theorem of Arithmetic illustrated and then assumed. (In Course 3 this principle may be proved.)

Course 2: Geometry, Measurement, and Probability

1. Intuitive Non-metric Geometry (25%)
2. Intuitive Metric Geometry (25%)
3. Probability (20%)
4. Further Geometry (20%)
5. The Real Number System (10%)

In Course 2 the system of positive rational numbers reappears in two new contexts: in the context of geometric measurement, where continuous sets are being compared, and in the context of probability where discrete sets are being "measured." Thus the system of positive rational numbers and the concept of measurement act as unifying threads. But the major blocks of new content covered are in geometry and probability.

Many opportunities for tying together these areas present themselves. For example, in the initial work in geometry which inevitably is concerned with establishing terminology and notation, combinatorial problems can be inserted to make the content more interesting. Later a probabilistic technique for approximating π could be given. Also, while studying probability, geometric representations can be given for many situations. For example, random processes are simulated by spinners, and experiments involving repeated trials are represented by random walks.

The geometry and the probability in this course are presented in a rather intuitive, non-deductive fashion. The main purpose here is to present the elementary facts in these areas, not to investigate their logical structure. Course 3 re-examines both areas from a more rigorous, deductive point of view.

The field of real numbers is also discussed to some extent in this second course.

1. Intuitive Non-metric Geometry (25%)

Geometry viewed as the study of subsets of an abstract set, called space, whose elements are called points; the subsets are called geometric figures; drawing conventional pictures of points, lines, and planes suggests incidence relations. Some of the logical connections between various incidence properties explored. (The 4-point geometry might be introduced here, but axiomatics should not receive much emphasis in this course.) The standard terminology associated with incidence (collinear, coplanar, concurrent, parallel, skew, ...) reviewed in a combinatorial context (e.g., into how many pieces is the plane partitioned by n lines no three of which are concurrent and no two of which are parallel?). Further geometric figures (half lines; rays; open, closed, and half-open segments; half planes, half spaces, plane and dihedral angles and their interiors and exteriors) defined in terms of the basic figures--points, lines, planes; the set operations; the intuitively presented notions of (arcwise) connectivity and betweenness. The standard notation for these figures reviewed, using combinatorial problems for motivation (e.g., how many angles are "determined" by n points no three of which are collinear?).

The ideas of polygonal and non-polygonal curves, simple curves, simple closed curves, and the interior of a simple closed curve presented intuitively along with a few other topological and geometrical concepts such as dimension of a figure, boundary of a figure, convexity; polygons, polyhedra, and Euler's formula.

Congruence introduced intuitively in this non-metric setting as meaning same size and shape. At this first contact, the notions of measurement and distance avoided. The natural development of ideas seems to be: congruence which leads to a process of measuring which in turn suggests the existence of a distance function. Perhaps here, but probably more appropriately in Course 3, a definition of congruence in terms of distance can be given. Congruence of segments, angles, and other plane and spatial figures, with perpendicularity introduced in terms of congruence of adjacent angles.

Congruence in the plane viewed in terms of intuitive notions of rigid motions of the plane (slides, turns, flips, and their compositions). Symmetries of figures in terms of invariant point sets and rigid motions. The composition of rigid motions is a rigid motion and the inverse of a rigid motion is a rigid motion. The group concept introduced to tie together algebra and geometry. A fuller treatment of transformation geometry is suggested in Course 4.

2. Intuitive Metric Geometry (25%)

The process of measuring described in terms of filling up the set to be measured with congruent copies of a unit and counting the number of units used. Illustrated for segments, angles, and certain plane and spatial figures. Integrally non-measurable figures (with

respect to a given unit) introduced and the positive rationals used as operators endowed with stretching-shrinking or replicating-partitioning powers. The rational number line reinterpreted in terms of segments and lengths, briefly showing the existence of rationally non-measurable segments and the real number line; a non-repeating infinite decimal exhibited and the theorem on decimal representation of irrationals recalled. The assignments of numbers to figures viewed as functions; observation that such measure functions are additive and invariant under congruence. The domain of segment measure functions extended to the domain of polygonal curves by additivity; perimeters computed. The additivity property applied to partitioning techniques for finding area; some familiar area and volume formulas derived (triangles, parallelograms, prisms, pyramids). The formula $A = \ell \times w$ for rectangles with irrational dimensions illustrated by drawing inscribed and circumscribed rectangles with rational dimensions. Plausible limiting arguments presented for circles and spheres; irrationality of π. The angle-sum theorem for triangles verified experimentally and then extended to convex n-gons by triangulation; the subsequent results about the various angle measures in regular polygons applied to making ruler-protractor drawings. Use of these measuring instruments suggests investigation of practical versus ideal measurement. Various units of length, area, angle, volume measurements and conversion factors relating them; the inevitability of approximation in practical measurement and the usage of such terms as "greatest possible error," "precision," "accuracy," " relative error." The various notational conventions in use for reporting how good an approximation is: significant digits, ± notation, interval of measure, scientific notation.

3. <u>Probability</u> (20%)

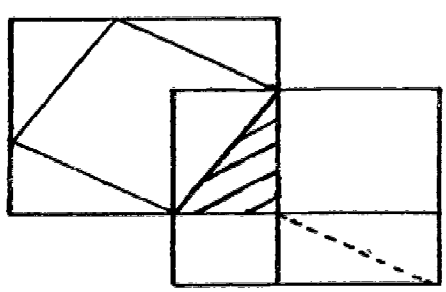
Various single and multi-stage experiments with discrete sample spaces considered and represented geometrically (trees, walks, spinners); large sample spaces and events "counted" using permutation and combination techniques. The terminology--sample space, outcome, event--compared with the terminology of geometry--space, point, figure; the assignment of probabilities to events compared with the assignment of lengths, areas, etc., to geometric figures. Both involve a comparison of two sets; the rational numbers are the indicated algebraic system. <u>A priori</u> assignment of probabilities (from shape of die, partitioning of spinner, constituency of urn, ...) compared with <u>a posteriori</u> assignment (long-range stability of relative frequency of events). The assignment of probabilities to events in terms of the point probabilities of their constituent outcomes, in the finite case, leading to additivity of probability measures; Venn diagrams used to illustrate the connection between set operations and logical connectives and to suggest the useful formulas,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{and} \quad P(A) = 1 - P(A') .$$

Conditional probability and independent events; problems involving multiplication along the branches of a tree. The connection between C^n_r and the number of paths of a certain kind in the plane; the formula $C^n_r\ p^r\ (1 - p)^{n-r}$ for r successes in n repeated trials deduced and used; Pascal's Triangle (if not described earlier). The work on probability should include many exercises, as this may be the first contact with the subject for many prospective elementary school teachers.

4. Further Geometry (20%)

The simplest straightedge-compass constructions reviewed and related to the parallel postulate and congruence conditions for triangles (some proving of triangles congruent appropriate here). Each straightedge-compass construction technique compared with a ruler-protractor drawing technique for the same figure. Some plausibility argument for the Pythagorean Theorem, perhaps the one suggested by this sketch.



The simplified congruence condition for right triangles stated. Some work on square roots is appropriate here: a proof, based on the Fundamental Theorem of Arithmetic, that $\sqrt{n}$ is irrational or a whole number; an algorithm or two for computing rational approximations to $\sqrt{n}$; a remark that the existence of square roots in R but not in Q is a tipoff that R must have some extra fundamental property that Q does not have; geometric construction of a segment with irrational length, with respect to a given unit. Projection techniques for drawing similar triangles reviewed. Applications to scale drawing and to straightedge-compass construction of the rational number line. Similarity conditions for triangles analogous to the congruence condi-

tions for triangles; for the case of right triangles the "AAA" similarity condition reduces to just "A". This suggests the calculation of trigonometric ratios and their use in indirect measurement. The Law of Cosines as a generalization of the Pythagorean Theorem.

5. The Real Number System (10%)

A brief review of the properties of the reals in some nonformal way--e.g., R includes Q; enjoys the same basic properties of +, ×, < that Q does but has one more property, namely, the least upper bound property. Use of this property to suggest but not prove the existence of $\sqrt[n]{a}$ for all $a \in R^+$ and all $n \in Z^+$. Some work with rational exponents.

Course 3: Mathematical Systems

1. The System of Whole Numbers (25%)
2. Fields (25%)
3. Geometry (25%)
4. Probability-Statistics (25%)

In this course certain portions of algebra, geometry, and probability are studied more deeply in a systematic, deductive way. Proof receives more emphasis than in Courses 1 and 2. Some of the concepts of logic itself receive explicit treatment, along with new results in algebra, geometry, and probability.

1. The System of Whole Numbers (25%)

The algebraic and order properties of W reviewed; the well-ordering principle introduced. Symbol $a|b$ defined; proofs of some simple divisibility theorems such as $a|b \Rightarrow a|bc$, $a|b$ and $a|c \Rightarrow a|b+c$. Various simple divisibility criteria of the base ten numeration system derived. Primes and prime factorization; the sieve of Eratosthenes; checking for prime divisors of n only up to $\sqrt{n}$; Euclid's Theorem and Wilson's Theorem. Unsolved problems such as Goldbach's conjecture and the twin primes problem. Other interesting odds and ends, e.g., figurate, deficient, abundant, and perfect numbers.

The important concepts of common and greatest common divisor (GCD) introduced and the existence, uniqueness, and linear combination expressibility theorems for GCD derived. Techniques for finding the GCD (by listing all divisors, by the Euclidean Algorithm, from known prime factorizations); the least common multiple (LCM), its existence and uniqueness proved, its relation to the GCD, and several techniques for finding it. (A flow diagram for the Euclidean Algorithm is appropriate here.) The concept of relative primeness and

the lemma stating that $p|ab \Longrightarrow p|a$ or $p|b$ (p prime), leading to a proof of the Fundamental Theorem of Arithmetic. Whether a rigorous proof of this theorem should be given using the well-ordering principle or mathematical induction, is debatable. The use of the Fundamental Theorem in reducing fractions and in demonstrating the existence of irrationals. Euler's ϕ-function might be defined and the theorems of Euler and Fermat illustrated.

2. Fields (25%)

Field defined using Q and R as prototypes. Subtraction and division in a field defined and their usual properties derived; Z_p (p prime) defined and shown to be a field; various properties of subtraction and division illustrated again in this context. Possibly enough group theory interposed (Lagrange's Theorem, order of element theorem) to prove the theorems of Euler and Fermat. In the context of solving a linear equation over a field, several concepts of logic can be studied; statement, equality, variable, open sentence, reference set, truth set; the unsolvability of $x^2 = 2$ over Q contrasts with its solvability over R; the completeness property of R recalled. The statement of this property depends on the concept of order; the definition of an ordered field abstracted from familiar properties of Q and R. Simple order properties deduced; Z_p shown to be unorderable (in any decent sense). The logical connectives and their relation to the set operations, within the context of solving inequalities over (say) R. Equivalence of open sentences; equivalence transformations. The Archimedean and density properties for R derived; a short excursion into limits (optional). The intermediate value theorem cited, behavior of polynomials for large $|x|$ illustrated; existence and uniqueness of positive n^{th} roots deduced.

3. Geometry (25%)

Incidence axioms suggested by the Euclidean plane and space stated as abstract axioms and exhibited in a finite model; simple incidence theorems proved and interpreted in both models. Illustr in the context of segments how the natural genesis of concepts: congruence (superposition) → process of measurement → distance function, can be reversed in a formalization of geometry: postulated distance function → congruence defined in terms of it. The intuitive notion of betweenness also defined in terms of distance; betweenness used to define rays and segments; the ruler postulate; proofs of a few elementary betweenness properties. (The depth to which this Birkhoff-SMSG approach is carried is a matter of taste. For the future elementary teacher it might be more appropriate to do most of the deductive work in the spirit of Euclid, pointing out from time to time an implicit betweenness or existence assumption.) Possible subjects for short deductive chains include: triangle congruence and straightedge-compass constructions; parallels, transversals, and angle sums; area postulates and an area proof of the Pythagorean Theorem.

Some flavor of other modern approaches to geometry, with attention restricted to the plane: coordinatization of the plane, vector addition and scalar multiplication of points, lines as subspaces and their cosets, vector and standard equations for lines; Pythagorean Theorem and its converse, Law of Cosines, perpendicularity, dot product, norm, distance; isometry, orthogonal transformation, matrix representation of linear transformations, classification of orthogonal transformations, decomposition of an isometry into a translation and an orthogonal transformation. Alternatively, a coordinate-free study of transformation groups.

4. <u>Probability-Statistics</u> (25%)

Review of the "natural" development of the terminology and basic concepts of outcome, sample space, event, point probability function, and probability measure; axioms for a probability measure. The possible backward rigorization in probability of the intuitive notion of "equally likely outcomes" as "outcomes having the same probability" compared with the backward rigorization in geometry of "congruent segments" as "segments having the same length." It might be worthwhile to digress in more generality on equivalence relations, partitions, functions, and preimages. After a few deductions from the axioms [$P(\emptyset) = 0$, $P(A') = 1 - P(A)$, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$], less emphasis is placed on deduction and further probabilistic concepts and techniques are presented. The idea of <u>simulation</u> by urn models, balls in cells, spinners, and the use of random number tables illustrated through a wide variety of problems. Permutations and combinations reviewed. More formal attention to combinatorial identities such as

$$2^n = (1+1)^n = \sum_{i=0}^{n} C_i^n, \qquad C_{k-1}^{n-1} + C_k^{n-1} = C_k^n,$$

$$\sum_{i=j}^{n-1} C_j^i = C_{j+1}^n .$$

More work on conditional probability, repeated trials, and random walks. Simple problems in hypothesis testing (e.g., ten tosses of a coin result in eight heads. With what confidence can you reject the hypothesis that the coin is honest?). Elementary expectation problems where expectation is thought of simply as a weighted average (e.g., how much should one expect to win on a roll of a die if the payoff when n turns up is n^2 dollars?). The same problem posed using a nonsymmetric spinner instead of a die. Common distribution functions; measures of spread.

Course 4: Functions

1. Real Functions (10%)
2. Algebraic Functions (20%)
3. Exponential and Logarithmic Functions (15%)
4. Transformations and Matrices (20%)
5. Trigonometric Functions (15%)
6. R^2 and the Dot Product (20%)

The purpose of this course is two-fold: to present a satisfying culmination to the four-course sequence and to prepare the student to continue in mathematics with a calculus course such as GCMC 1. Both of these goals are met in the context of a course centered around the function concept. Preparation for calculus is accomplished by studying special real functions, namely, the elementary functions; study of special functions of the plane (affine transformations) provides an appropriate dénouement of the four-course sequence by bringing together arithmetic, algebra, and geometry in a transformation approach to plane geometry.

1. Real Functions (10%)

Brief review of the general concept of function from both the rule-of-assignment and ordered-pair points of view; specialization to the case where the domain and range are real numbers. Simple examples of functions--some artificial, some from science or business. Graphing and reading graphs. Graphs of real functions make it easier to think of them as mathematical objects in their own right, subject to operations as are other mathematical objects. Addition, subtraction, multiplication, division, and composition of functions viewed graphically as well as algebraically. Various additive and multiplicative groups of functions. One could look for rings, vector spaces, or even algebras of functions if that much abstract algebra is available.

2. Algebraic Functions (20%)

Real functions specialized to polynomial functions with emphasis on linear and quadratic functions. Slope and equations of straight lines, zeros of polynomials, and the factor theorem. Graphical interpretation of linear functions in the plane; geometric interpretation of linear functions on the number line in terms of stretches, shrinks, slides, and flips. (This suggests considering analogous functions of the plane and presents a natural opening for the discussion of general mappings of the plane and then the special types of mappings which are important for geometry, namely, translations, rotations, magnifications, and reflections. A discussion of transformation geometry may be included at this point.)

Increasing and decreasing real functions; there is no analogous concept for plane functions since the plane is not ordered. The completeness of the real number system and the intermediate value theorem done at an intuitive level; invertible functions, both in the context of functions of the plane and in the context of real functions. For real functions this leads to work with roots and rational exponents. Quadratic equations and various explicit algebraic functions.

3. Exponential and Logarithmic Functions (15%)

It is not reasonable to give a rigorous development of exponential functions. After adequate study of a^x for x rational and after some geometric motivation, the existence of a^x for x real should be assumed. The "laws of exponents" need emphasis. Other isomorphisms and homomorphisms recalled. Graphs of exponential functions and combinations thereof; logarithm functions defined as inverses of exponential functions, their properties derived from the properties of exponential functions; a minimal amount of computational work with common logarithms.

4. Transformations and Matrices (20%)

Having just completed computational work with some real functions, one can naturally ask whether various functions of the plane can also be given a concrete numerical representation. This leads to linear algebra and the study (in R^2) of vectors, dependence, independence, basis, linear transformation, matrix representation of linear transformations, matrix multiplication, and transformation composition.

5. Trigonometric Functions (15%)

The sine and cosine functions introduced by recalling the trigonometric ratios (Course 2); their definitions in terms of the winding function. The other trigonometric functions defined, graphs drawn, and questions of periodicity and invertibility entertained. Rotation matrix derivation of the addition formulas for sines and cosines. Proof of other trigonometric identities. The Pythagorean Theorem and its converse recalled and the law of cosines proved as a generalization.

6. R^2 and the Dot Product (20%)

The dot product motivated by the law of cosines as a measure of perpendicularity. The chain of ideas from dot product through length to metric traced. The central role played by this metric in currently popular axiomatic developments of geometry. The geometric and algebraic significance of the determinant function for 2 x 2 matrices.

Glossary of Symbols

SYMBOL	MEANING
W	$\{0, 1, 2, \ldots\}$, the set of whole numbers
Z	The set of integers
Q	The set of rational numbers
R	The set of real numbers
C	The set of complex numbers
Z^+, Q^+, R^+	The set of positive elements of Z, Q, R, respectively
Z_n	The set of integers modulo n
$x \in A$	The element x belongs to the set A
$A \subset B$	The set A is a subset of set B
$A \times B$	$\{(x,y) \mid x \in A \text{ and } y \in B\}$
A^2	$A \times A$
A^3	$A \times A \times A$
$\Longrightarrow$	implies
$a \mid b$	a divides b
GCD	Greatest common divisor
LCM	Least common multiple
C^n_r	$\frac{n!}{r!(n-r)!}$
P^n_r	$\frac{n!}{(n-r)!}$
2^A	The set of all subsets of A
$[x]$	The set of all polynomials in one indeterminate with coefficients in A

APPENDIX II. COURSE GUIDES FOR GEOMETRY

INTRODUCTION

Euclidean geometry will, of course, continue to be taught in our schools, but there is also a tendency toward basing the teaching of geometry on linear algebra. A fundamental reason for this is that such a course reveals the unity of algebra and geometry in a way that the Euclidean approach does not.

It is our belief that both approaches have sufficient interest for the prospective mathematics teacher that study of each should be required. We therefore include a suggested outline for each kind of course. As with the Level I Course Guides our intention here is not to be prescriptive but to present outlines which might be useful for interested persons in devising appropriate courses. For that reason, and also because it is newer, we have gone into more detail with the algebraically oriented course.

Mathematics 9: Foundations of Euclidean Geometry (3 semester hours)

The purpose of this course is two-fold. On the one hand it presents an adequate axiomatic basis for Euclidean geometry, including the one commonly taught in secondary schools, while on the other hand it provides insight into the interdependence of the various theorems and axioms. It is this latter aspect that is of the greater importance for it shows the prospective teacher that there is no one Royal Road to the classical theorems. This deeper appreciation of geometry will better prepare the teacher to assess the virtues of alternative approaches and to be receptive to the changes in the secondary school geometry program that loom on the horizon.

Courses similar to this have now become commonplace. As a consequence, no great detail should be necessary in this guide. There is a greater abundance of appropriate topics than can be covered in one course, so some selection will always need to be made.

Although enough consideration should be given to three-space to build spatial intuition, the major emphasis should be on the plane, since it is in two-space that the serious and subtle difficulties first become apparent. The principal defects in Euclid's Elements relate to the order and separation properties and to the completeness of the line. Emphasis should be directed to clarifying these subtle matters with an indication of some of the ways by which they can be circumvented. The prospective teacher must be aware of these matters and have enough mathematical sophistication to proceed to new topics with only an indication of how they are resolved.

The course consists of six parts, after brief historical introduction and a critique of Euclid's Elements. The allotment of times that have been assigned for these parts are but suggestions to be used as a guide, because emphasis will vary with the background of the students, the text used, and the tastes of the instructor. Prerequisites for the course are a modest familiarity with rigorous deduction from axioms, for example as encountered in algebra, and the completeness of the real number system.

1. Incidence and Order Properties (8 lessons)

In this part of the course, after a brief treatment of incidence properties, the inherent difficulties of betweenness and separation are discussed. The easiest, and suggested, way to proceed is in terms of distance. The popular method today is to use the Birkhoff axioms, or a modification such as given by SMSG. In addition, one should give some indication of a synthetic foundation for betweenness such as that of Hilbert. A brief experience with a synthetic treatment of betweenness is enough to convince the student of the power of the metric apparatus.

Alternatively, one can begin with a synthetic treatment of betweenness and then introduce the metric apparatus. With this approach, metric betweenness is a welcome simplification.

2. Congruence of Triangles and Inequalities in Triangles (8 lessons)

It is recommended that angle congruence be based on angle measure (the Birkhoff axioms). Yet here too some remarks on a synthetic approach are desirable.

The order of presentation of the congruence theorems can depend on the underlying axiom system used. What is perhaps more important is to observe their interrelations. At this point a global view of transformations of the plane should receive attention. Ruler and compass constructions should be deferred, as the treatment is simpler and more elegant after the parallel axiom has been introduced. The triangle inequality and the exterior angle theorem occur here.

3. Absolute and Non-Euclidean Geometry (6 lessons)

Up to this point there has been no mention of the parallel postulate. It is desirable to explore some of the attempts to prove it. One should prove a few theorems in absolute geometry, in particular ones about Saccheri quadrilaterals. Then some theorems in hyperbolic geometry can be given, among which the angle-sum theorem for angles in a triangle is most important. A model, without proof, for hyperbolic geometry is natural here.

This part of the course can also be taught after Part 4 when Euclid's parallel axiom and consequences of it have been covered.

4. The Parallel Postulate (8 lessons)

There are many topics, of central importance in high school, that need to be discussed in this part of the course. It is desirable to give here, as well as in Part 3, considerable attention to the history of the parallel axiom. Due to time limitations, it will probably be necessary to omit some topics. Nevertheless, some attention should be given to: parallelograms, existence of rectangles, Pythagorean Theorem, angle-sum theorem for triangles, similarity, ruler and compass constructions, and an introduction to the notion of area.

5. The Real Numbers and Geometry (8 lessons)

This part is devoted to matters in which the completeness of the real number system plays a role. Some attention must be given to the completeness of the line and the consequences thereof. Archimedes' axiom arises naturally here. Important topics are: similarity of triangles for the incommensurable case; circumference; area in general and, in particular, area of circles; and, finally, a coordinate model of Euclidean geometry. It is possible to give a coordinate model of a non-Archimedean geometry at this time.

6. Recapitulation (3 lessons)

This part is intended to give perspective on the preceding sections. It should have a strong historical flavor and might well include lectures with outside reading or a short essay.

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Prenowitz, W. and Jordan, M. Basic Concepts of Geometry. New York, Blaisdell, 1965.

Mathematics 9a: Vector Geometry

There are approaches to geometry other than the classical synthetic Euclidean approach and several of these are being suggested for use in both the high school and college curricula. Moreover, exposure to different foundations for geometry yields deeper insights into geometry and can serve to relate Euclidean geometry to the mainstream of current mathematical interest. It is this latter reason which underlies much of the discussion about geometry that is now prevalent. There are at least three approaches that merit consideration.

I. The classical approach of Felix Klein, wherein one begins with projective spaces and, by considering successively smaller subgroups of the group acting on the space, one eventually arrives at Euclidean geometry. A course of this nature might be called projective geometry, but it should proceed as rapidly as possible to Euclidean geometry. Besides books on projective geometry, other references are:

1. Artin, Emil. Geometric Algebra. New York, Interscience, 1957.

2. Gans, David. Transformations and Geometries. New York, Appleton-Century-Crofts, Inc., 1968.

3. Klein, Felix. Vorlesungen Über Nicht-Euklidische Geometrie. New York, Chelsea Publishing Company, 1959.

4. Schreier, Otto and Sperner, Emanuel. Projective Geometry of n Dimensions. New York, Chelsea Publishing Company, 1961.

(Throughout this outline, references are given because of their content with no implication that the level of presentation is appropriate. Indeed, adjustments will normally be necessary.)

II. The transformation approach, which in some ways is a variant of Klein's, uses the Euclidean group to define congruence and other familiar concepts. As a further variant of this, there are treatments which begin with synthetic Euclidean geometry and proceed to the Euclidean group. References are:

5. Bachmann, F. Aufbau der Geometrie aus dem Spiegelungsbegriff. Berlin, Springer-Verlag, 1959.

6. Choquet, Gustave. Geometry in a Modern Setting. Boston, Massachusetts, Houghton Mifflin Co., 1969.

7. Coxford, A. F. and Usiskin, Z. P. Geometry, A Transformation Approach, Vol. I, II. River Forest, Illinois, Laidlaw Brothers, 1970.

8. Eccles, Frank. An Introduction to Transformational Geometry. Reading, Massachusetts, Addison-Wesley Publishing Co., Inc., 1971.

III. The vector space approach, the one suggested for this course, uses vector spaces as an axiomatic foundation for the investigation of affine and Euclidean geometry. Through the use of vector spaces, classical geometry is brought within the scope of the central topics of modern mathematics and, at the same time, is illuminated by fresh views of familiar theorems. Some of the references below contain isolated chapters which are relevant to this approach; in such cases these chapters are indicated.

9. Artin, Emil. Geometric Algebra. New York, Interscience, 1957.

10. Artzy, Rafael. Linear Geometry. Reading. Massachusetts, Addison-Wesley Publishing Co., Inc., 1965.

11. Dieudonné, Jean. Linear Algebra and Geometry. Boston, Massachusetts, Houghton Mifflin Co., 1969.

12. Gruenberg, K. W. and Weir, A. J. Linear Geometry. New York, Van Nostrand-Reinhold Books, 1967.

13. MacLane, Saunders and Birkhoff, Garrett. Algebra. New York, Macmillan Co., 1967. (Chapters VII, XI, XII)

14. Mostow, George; Sampson, Joseph; Meyer, Jean-Pierre. Fundamental Structures of Algebra. New York, McGraw-Hill Book Co., 1963. (Chapters 8, 9, 14)

15. Murtha, J. A. and Willard, E. R. Linear Algebra and Geometry. New York, Holt, Rinehart and Winston, Inc., 1969.

16. Snapper, Ernst and Troyer, Robert. Metric Affine Geometry. New York, Academic Press, 1971.

The course outlined below has as prerequisite an elementary course in linear algebra (GCMC 3). The main topics are:

1. Affine Geometry and Affine Transformations
2. Euclidean Geometry and Euclidean Transformations
3. Non-Euclidean Geometries.

Because of the relative unfamiliarity of this approach to geometry, more details such as definitions and typical results will be included. Also, a brief justification is given.

In Euclidean geometry one considers the notion of a translation of the space into itself. These translations form a real vector space under the operation (addition) of function composition and multiplication by a real number. Thus the "vector space of translations" acts on the set of points of Euclidean space and satisfies the following two properties.

A. If (x,y) is an ordered pair of points, there is a translation T such that $T(x) = y$. Moreover, this translation is unique.

B. If T_1 and T_2 are translations and x is a point, then the definition of "vector addition" as function composition is indicated by the formula

$$(T_1 + T_2)(x) = T_1(T_2(x)).$$

With this intuitive background, the details of the course outline will now be given. The definitions and propositions are stated for dimension n, since this causes no complication, but the emphasis will be on dimensions two and three.

1. Affine Geometry and Affine Transformations

One defines real n-dimensional affine space as the triple (V, X, μ) where V is a real vector space of dimension n (the vector space of the translations), X is the set of points of the geometry and μ: $V \times X \to X$ defined by $\mu(T,x) = T(x)$ is the action of V on X which satisfies properties A and B above. For convenience, the affine space (V, X, μ) is usually denoted simply by X.

Affine subspaces of X are defined as follows. Let $x \in X$ and let U be a linear subspace of V (a subspace of translations). The affine subspace determined by x and U is denoted by $S(U, x)$ and consists of the set of points

$$\{T(x) \mid T \in U\},$$

i.e., $S(U,x)$ consists of all translates of x by a translation belonging to U. The dimension of $S(U,x)$ is defined to be the dimension of U. Then one-dimensional affine subspaces are called lines, two-dimensional affine subspaces are called planes and (n-1)-dimensional affine subspaces are called hyperplanes (n = dimension of V).

Two affine subspaces S and S' are called parallel ($S \parallel S'$) if there exists a translation T such that $T(S) \subset S'$ or $T(S') \subset S$.

Parallelism and incidence are investigated, with special emphasis on dimensions two and three. Such results as the following are obtained.

a. Lines ℓ and m in the plane are parallel if and only if $\ell = m$ or $\ell \cap m = \emptyset$.

b. A line ℓ and a plane π in three-space are parallel if and only if $\ell \subset \pi$ or $\ell \cap \pi = \emptyset$. If $\ell \not\parallel \pi$, then $\ell \cap \pi$ is a point.

c. There exist skew lines in three-space.

d. Planes π and π' in three-space are parallel if and only if $\pi = \pi'$ or $\pi \cap \pi' = \emptyset$. If $\pi \not\parallel \pi'$, then $\pi \cap \pi'$ is a line.

A coordinate system for the affine space X consists of a point $c \in X$ and an ordered basis for V. A point $x \in X$ is assigned the coordinates $(x_1, \ldots, x_n)$ if T is the unique translation such that $T(c) = x$ and T has coordinates $(x_1, \ldots, x_n)$ with respect to the given ordered basis for V. Using these notions, one can study analytic geometry, e.g., the parametric equations for lines, the linear equations for hyperplanes, the relationship between the linear equations of parallel hyperplanes, incidence in terms of coordinate representations, etc.

For each point $c \in X$, there is a natural way to make X into a vector space which is isomorphic to V. Namely, if r is a real number, $x,y \in X$, and T_1, T_2 are the unique translations satisfying $T_1(c) = x$ and $T_2(c) = y$, then one defines

$$x + y = T_2(T_1(c)) \quad \text{and} \quad rx = (rT_1)(x).$$

The vector space X_c with origin c is the tangent space of classical differential geometry. (Affine space is often defined as the vector space V itself; this approach to affine geometry is based on the isomorphism between X_c and V.)

An affine transformation is a function $f: X \to X$ with the following properties:

a. f is one-to-one and onto.

b. If ℓ and ℓ' are parallel lines, then $f(\ell)$ and $f(\ell')$ are parallel lines.

The affine transformations form a group called the affine group which contains the translation group as a commutative subgroup. For each point $c \in X$, the set of affine transformations which leave c fixed form a subgroup of the affine group; moreover, this subgroup is the general linear group of the vector space X_c and is therefore isomorphic to the general linear group of V. Finally, properties of affine transformations are investigated.

Other topics of affine geometry which are studied include orientation, betweenness, independence of points, affine subspace spanned by points, and simplexes.

2. Euclidean Geometry and Euclidean Transformations

Euclidean space is defined as the affine space (V, X, μ) where V has been given the additional structure of a positive-definite inner product. Thus for each $T \in V$, T^2 is a non-negative real number. A distance function is introduced on X by defining the distance between an ordered pair (x,y) of points of X to be $\sqrt{T^2}$ where T is the unique translation such that $T(x) = y$. A Euclidean transformation (rigid motion, isometry) of X is a mapping of X which preserves distance.

The Euclidean transformations form a subgroup of the affine group. For each $c \in X$, the Euclidean transformations which leave c fixed form a subgroup of the Euclidean group. In fact, this is the orthogonal group of the vector space X_c (with the inner product induced on it from V through the given isomorphism) and therefore is isomorphic to the orthogonal group of V.

Rotations and reflections are first defined for the Euclidean plane and then for n-dimensional space. The Cartan-Dieudonné theorem becomes an important tool in the investigation of the Euclidean group. It states that every Euclidean transformation of n-space is the product of at most n + 1 reflections in hyperplanes. It follows immediately that there are four kinds of Euclidean transformations of the Euclidean plane: translations, rotations, reflections, and glide reflections.

Rotations and reflections of Euclidean three-space are investigated. From the Cartan-Dieudonné theorem it follows that every rotation of three-space has a line of fixed points (the axis of rotation). The set of all rotations with a given line ℓ as axis is a subgroup of the rotation group of three-space. Moreover, this rotation group with axis ℓ is isomorphic to the rotation group of the Euclidean plane, thus giving the classical result that every rotation of three-space is determined by an axis and a given "angle of rotation."

One now defines a figure to be a subset of X and calls two figures congruent if there is a Euclidean transformation which maps one figure onto the other. Using these concepts, one proceeds to proofs of the classical congruence theorems of plane geometry (S.S.S., S.A.S., A.S.A., H.S.).

Finally, orthogonality and similarity are investigated.

3. Non-Euclidean Geometries

The classical method of obtaining a non-Euclidean plane geometry is to replace the parallel postulate by another postulate on parallel lines and thus obtain hyperbolic geometry. Here the approach is different. The positive-definite inner product is replaced by other (nonsingular) inner products. The geometry obtained is non-Euclidean, but the parallel postulate is still valid! This startling result is true because the underlying space is the affine plane (in which the parallel postulate is valid) and the change of inner product does not disturb the affine structure.

Actually, the investigation of non-Euclidean geometries can be made concurrently with that of Euclidean geometry. For example, the Lorentz plane and the negative Euclidean plane can be defined and investigated at the same time as the Euclidean plane. "Circles" in the Lorentz plane are related to hyperbolas of the Euclidean plane, etc.

One of the major results is Sylvester's theorem, from which one concludes that there are precisely n + 1 distinct nonsingular geometries which can be placed on n-dimensional affine space.